

EXTREMAL PERMUTATIONS IN ROUTING CYCLES

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ABSTRACT. Let G be a graph on n vertices, labeled v_1, \dots, v_n and π be a permutation on $[n] := \{1, 2, \dots, n\}$. Suppose that each pebble p_i is placed at vertex $v_{\pi(i)}$ and has destination v_i . During each step, a disjoint set of edges is selected and the pebbles on each edge are swapped. Let $rt(G, \pi)$, the routing number for π , be the minimum number of steps necessary for the pebbles to reach their destinations.

Li, Lu, and Yang prove that $rt(C_n, \pi) \leq n - 1$ for any permutation on n -cycle C_n and conjecture that for $n \geq 5$, if $rt(C_n, \pi) = n - 1$, then $\pi = (123 \cdots n)$ or its inverse. By a computer search, they show that the conjecture holds for $n < 8$. We prove in this paper that the conjecture holds for all even n .

1. INTRODUCTION

Routing problems occur in many areas of computer science. Sorting a list involves routing each element to the proper location. Communication across a network involves routing messages through appropriate intermediaries. Message passing between multiprocessors requires the routing of signals to correct processors.

In each case, one would like the routing to be done as quickly as possible. We will use a routing model first introduced by Alon, Graham, and Chung [2] in 1994. Let $G = (V, E)$ be a graph. Label the vertices as v_1, \dots, v_n and each vertex sits with a pebble. Suppose that under permutation π on $[n]$, a pebble p_i is placed at $v_{\pi(i)}$. We wish to move pebbles to their destinations. To do so, we select a matching of G and swap the pebbles at the endpoints of each edge and repeat in next round until all pebbles are in places.

Let $rt(G, \pi)$ denote the minimum number of rounds necessary to route π on G . Then, the **routing number** of G is defined as:

$$rt(G) = \max_{\pi} \{rt(G, \pi)\}$$

As the routing problem occurs in problems in computer science, some of the first bounds shown are consequences of computer science algorithms. The odd-even transposition sort [4] and Benes network [3] show $rt(P_n) = n$ and $rt(Q_n) \leq n - 1$, for path P_n of n -vertices and n -dimensional hypercube Q_n , respectively.

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Very few results are known for the exact values of the routing numbers of graphs. Alon, Chung, and Graham [2] prove

Theorem 1 (Alon, Chung, and Graham [2]). (1) $rt(K_n) = 2$ and $rt(K_{n,n}) = 4$;
(2) $rt(G) \geq \text{diam}(G)$ and $rt(G) \geq \frac{2}{|C|} \min\{|A|, |B|\}$, where $\text{diam}(G)$ is the diameter of G and C is a set that cuts G into parts A and B ;
(3) $rt(G) \leq rt(H)$ and $rt(T_n) < 3n$, where H is a spanning subgraph of G and T_n is a tree on n vertices;
(4) $rt(G_1 \times G_2) \leq 2rt(G_1) + rt(G_2)$, and $n \leq rt(Q_n) \leq 2n - 1$.

Zhang [6] improves their bound on trees, showing $rt(T_n) \leq \lfloor \frac{3n}{2} \rfloor + O(\log n)$.

Li, Lu, and Yang [5] show $n + 1 \leq rt(Q_n) \leq 2n - 2$, improving both the previous upper and lower bounds on hypercubes. Among other results, they also give the exact routing number of cycles: $rt(C_n) = n - 1$. Furthermore, they made following conjecture.

Conjecture 1 (Li, Lu, Yang [5]). For $n \geq 5$, if $rt(C_n, \pi) = n - 1$, then π is the rotation $(123 \cdots n)$ or its inverse.

The conjecture does not hold for $n = 4$; the permutation that transposes two non-adjacent vertices and fixes the other two serves as a counterexample. They verified the conjecture for $n < 8$ through a computer search. The conjecture hints towards a very counter-intuitive idea, that the worst case permutation on the cycle is one where each pebble is only distance one away from its destination.

In this article, we give a proof of the conjecture when n is even.

Theorem 2. For even $n \geq 6$, if $rt(C_n, \pi) = n - 1$, then π is the rotation $(123 \cdots n)$ or its inverse.

It worths to note that some new tools are introduced in the proof, beyond the ideas from [1] by Albert, Li, Strang, and the last author. Those tools are introduced in the Section 2. In Section 3, we present a few important lemmas; in Section 4, we discuss the possible extremal situations, and in Section 5, we discuss how to deal with the extremal situations.

2. A FEW IMPORTANT NOTION AND TOOLS

2.1. Spins and Disbursements. Let $G = C_n$ and label the vertices of C_n as v_1, v_2, \dots, v_n in a clockwise order. Let the clockwise direction be the positive direction and counter clockwise be the negative direction. There are exactly two paths for pebble p_i to reach its destination, either by traveling in the positive or negative direction. Let $d^+(v_i, v_j)$ denote the distance from v_i to v_j when traveling along the cycle in the positive direction and $d^-(v_i, v_j)$ the distance when traveling in the negative direction. Note that $d^+(v_i, v_j) + d^-(v_i, v_j)$ equals n when $i \neq j$ and 0 when $i = j$. For simplicity, for pebbles p_i and p_j , we define $d^+(p_i, p_j) = d^+(v_{\pi(i)}, v_{\pi(j)})$.

Consider a routing process of π on C_n with pebble set $P = \{p_1, \dots, p_n\}$. For each pebble p_i , let $s(p_i)$, the *spin* of p_i , represent the displacement for p_i to reach its destination from its current position. So, $s(p_i) \in \{d^+(v_{\pi(i)}, v_i), d^+(v_{\pi(i)}, v_i) - n\}$. The sequence $B = (s(p_1), s(p_2), \dots, s(p_n))$ is called a **valid disbursement** of π . The disbursement describes the direction in which each pebble will move in the routing. Note that the spin of a pebble changes with its movement.

Not all possible combinations of spins produce valid disbursements. The following lemma gives a necessary and sufficient condition for a set of spins to be a valid disbursement.

Lemma 1. *Let $(s(p_1), s(p_2), \dots, s(p_n))$ be an assignment of the spins to the pebbles. It is a valid disbursement if and only if $\sum_{p \in P} s(p) = 0$.*

Proof. To see the necessity, we observe that when two pebbles are swapped, one moves forward one step and one moves backward one step, so the sum of spins stays. As B is a valid disbursement, the final spins are all zeroes, so the sum is also zero.

For sufficiency, we can move the pebbles one by one along their assigned directions. \square

From this lemma we know that there is at least one pebble p_i with positive spin and one pebble p_j with negative spin in a valid disbursement if π is not identity. If we change the spins of p_i and p_j so that they move in the opposite directions, the new spins still give a valid disbursement. We say that we **flip the spins of p_i and p_j** when we apply this change.

A valid disbursement $(s(p_1), \dots, s(p_n))$ is minimized if $\sum_{p \in P} |s(p)|$ is minimized. The following simple fact is very important.

Lemma 2. *If a valid disbursement is minimized, then $s(p_i) - s(p_j) \leq n$ for all $i, j \in [n]$.*

Proof. For otherwise we would flip the spins to make the sum smaller. \square

It is not hard to show the converse is also true, so one can apply the flips on a valid disbursement to get a minimized disbursement. We omit the proof here.

2.2. An order relation. It is clear to see that if $s(p_i) - s(p_j) > d^+(p_i, p_j)$, then p_i and p_j will swap at some round in the routing process. For that purpose, we define the following order relation on pebbles.

Definition 1. *Given a disbursement B , we call $p_i \succ p_j$ if $s(p_i) - s(p_j) > d^+(p_i, p_j)$*

Remark: when we mention order of pebbles in the text, by default it is always associated with the current disbursement.

Note that the order relation is transitive, for if $p_i \succ p_j$ and $p_j \succ p_k$, then $s(p_i) - s(p_j) > d^+(p_i, p_j)$ and $s(p_j) - s(p_k) > d^+(p_j, p_k)$, and it follows that $s(p_i) - s(p_k) > d^+(p_i, p_j) + d^+(p_j, p_k) \geq d^+(p_i, p_k)$, so $p_i \succ p_k$.

As two pebbles have different destinations, $s(p_i) - s(p_j) \neq d^+(p_i, p_j)$, so if $p_i \succ p_j$ is not true, then $s(p_i) - s(p_j) < d^+(p_i, p_j)$. When $p_i \succ p_j$, we sometimes call p_i **is bigger than** p_j and p_j **is smaller than** p_i . If p_i is neither bigger nor smaller than p_j , we call them **incomparable**. If all pebbles in set P_1 are bigger than all pebbles in P_2 , we also write $P_1 \succ P_2$.

The following lemma provides a convenient way to determine order relations.

Lemma 3. *Let x, y, z be three pebbles in the clockwise order sitting on the cycle. If $x \succ z$, then $x \succ y$ or $y \succ z$. Furthermore, if $x \succ z$, then y is not smaller than z .*

Proof. For otherwise, $s(x) - s(y) < d^+(x, y)$ and $s(y) - s(z) < d^+(y, z)$. It follows that $s(x) - s(z) < d^+(x, y) + d^+(y, z) = d^+(x, z)$, thus x is not bigger than z , a contradiction.

For the furthermore part, if $x \succ z$ and $z \succ y$, then $s(x) - s(z) \geq d^+(x, z)$ and $s(z) - s(y) \geq d^+(z, y)$, and it follows that $s(x) - s(y) \geq d^+(x, z) + d^+(z, y) > n$, a contradiction. \square

The following lemma says that it is enough to swap two comparable pebbles to route the permutation.

Lemma 4. *Assume B is a minimized disbursement of π . If pebble p is incomparable with all other pebbles, i.e., there exists no pebble q such that $q \succ p$ or $p \succ q$, then $s(p) = 0$, i.e., pebble p has arrived its own destination vertex.*

Proof. Suppose the pebble p is incomparable to other pebbles and $s(p) \neq 0$. By symmetry we assume that $s(p) > 0$.

Let $\pi = \Pi_i \pi_i$ be a cycle decomposition of π , where $\pi_i = (i_1, \dots, i_{r_i})$, i.e., the pebble placed at v_{i_k} and has destination $v_{i_{k+1}}$ for all $k \leq r_i$, with $i_{r_i+1} = i_1$. Let P_i be the set of pebbles on π_i , and we call π_i to be an orbit of those pebbles.

We claim that for each orbit P_i , $\sum_{q \in P_i} s(q) = an$ for some integer a . To see this, we note that $s(p_{i_k}) \in \{d^+(v_{i_k}, v_{i_{k+1}}), d^+(v_{i_k}, v_{i_{k+1}}) - n\}$. Thus if all spins are positive, we would have the sum to be bn for some positive integer b . However, each switch of the spin from positive to negative would cause a change of $-n$ in the sum. So the sum of spins stays as a multiple of n .

Assume $p = p_{i_1}$ is a pebble of P_i . We claim that there exists no pebble in P_i will pass v_{i_2} in the negative direction to arrive its destination. Otherwise, assume p_{i_k} ($2 < k \leq r_i$) is such a pebble, then we have $s(p_{i_k}) + d^+(v_{i_2}, p_{i_k}) < 0$, notice that p_{i_1} is placed at v_{i_2} , thus we have

$$s(p_{i_1}) > 0 > s(p_{i_k}) + d^+(v_{i_2}, p_{i_k}) = s(p_{i_k}) + d^+(p_{i_1}, p_{i_k}),$$

hence $p = p_{i_1} \succ p_{i_k}$, a contradiction.

Furthermore, we have $s(p_{i_2}) > 0$. Otherwise, if $s(p_{i_2}) < 0$, notice that the destination of p_{i_2} is v_{i_2} where p_{i_1} been placed, thus we have $s(p_{i_2}) = d^+(p_{i_2}, p_{i_1}) - n = -d^+(p_{i_1}, p_{i_2})$, hence

$$s(p_{i_1}) > 0 = s(p_{i_2}) + d^+(p_{i_1}, p_{i_2})$$

and it follows that $p = p_{i_1} \succ p_{i_2}$, a contradiction.

The fact $s(p_{i_1}) > 0$ implies that p_{i_1} will travel from v_{i_2} in the positive direction, and the fact $s(p_{i_2}) > 0$ implies that p_{i_2} will arrive to its destination v_{i_2} in the positive direction, since there exists no pebble will pass v_{i_2} in the negative direction, therefore we have $\sum_{q \in P_i} s(q) = bn$ for some positive integer b .

As the sum of all spins is zero, there must exists some orbit P_j with spin sum cn for some integer $c < 0$. In particular, there exists a pebble $q \in P_j$ such that q passes p in the negative direction to arrive its destination. So $s(q) + d^+(p, q) < 0 < s(p)$ and it follows that $p \succ q$, a contradiction. \square

By above lemma, in our routing process, we will only swap comparable pebbles. The following lemma says that whether two pebbles swap is determined by the initial disbursement. So we will not keep track of the spins, but just see whether necessary swaps are swapped.

Lemma 5. *If $p_i \succ p_j$, then after swap p_i and p_j are incomparable. If p_i and p_j are incomparable, then in the sorting process, they will be always incomparable.*

Proof. Suppose that $p_i \succ p_j$ and after swap of p_i and p_j , $p_j \succ p_i$. Then $n \geq s(p_i) - s(p_j) \geq d^+(p_i, p_j) + 1 \geq 2$ and right after the swap of p_i and p_j , $s(p_i) - s(p_j)$ reduces at most $d^+(p_i, p_j)$, thus still positive. Let $s'(p_i)$ and $s'(p_j)$ be the new displacement of p_i and p_j , respectively. then $s'(p_i) - s'(p_j) > 0$. Therefore $s'(p_j) - s'(p_i) < 0$, and p_j cannot be bigger than p_i ; also, $s'(p_i) - s'(p_j) \leq n - 2$ and less than the distance $n - 1$ from p_i to p_j , thus after the swap, p_i cannot be still bigger than p_j .

If p_i and p_j are incomparable, then in the sorting process, $(s(p_i) - s(p_j)) - d^+(p_i, p_j)$ does not change: if a pebble swaps with both (the distance does not change and $s(p_i) - s(p_j)$ does not change, if a pebble swaps only with p_i , then $s(p_i)$ increases by one and $d^+(p_i, p_j)$ increases by one, if a pebble swaps only with p_j , then $s(p_j)$ increases by one and $d^+(p_i, p_j)$ reduces by one. \square

When B is minimized and two pebbles p_i and p_j satisfy $s(p_i) - s(p_j) = n$, then after flip the spins of p_i and p_j we still get a minimized disbursement. The following lemma tells us the change of the order relation when we do such a flip.

Lemma 6. *Let B be a minimized disbursement of π , and $s(p_i) - s(p_j) = n$. If flip the spins of p_i to $s'(p_i) = s(p_i) - n$ and p_j to $s'(p_j) = s(p_j) + n$, then for $k, l \notin \{i, j\}$ we have*

- (1) $p_j \succ p_i$; and the order relation remains unchanged for p_k and p_l .
- (2) $p_k \succ p_i$ if and only if before the flip p_i and p_k were incomparable. Similarly, $p_j \succ p_l$ if and only if before the flip p_j and p_l were incomparable.
- (3) p_i and p_k are incomparable if before the flip $p_i \succ p_k$. Similarly, p_i and p_l are incomparable if before the flip $p_l \succ p_j$.

Proof. (1) As $s'(p_j) - s'(p_i) = (s(p_j) + n) - (s(p_i) - n) = 2n - (s(p_i) - s(p_j)) = n > d^+(p_j, p_i)$, $p_j \succ p_i$ after the flip. For $k, l \notin \{i, j\}$, the spins of p_k, p_l and the distance $d^+(p_k, p_l)$ do not change by flipping the spins of p_i and p_j , so their order relation does not change as well.

(2) For $k \notin \{i, j\}$, we know that

$$s(p_k) - s'(p_i) - d^+(p_k, p_i) = s(p_k) - (s(p_i) - n) - (n - d^+(p_i, p_k)) = s(p_k) - s(p_i) + d^+(p_i, p_k).$$

Note that p_k cannot be bigger than p_i before the flip, for otherwise, $s(p_k) - s(p_j) > s(p_i) - s(p_j) = n$, a contradiction, and $s(p_i) \neq s(p_k) + d^+(p_i, p_k)$ as pebbles must have different destinations. Therefore we have

$$s(p_k) - s'(p_i) > d^+(p_k, p_i) \text{ if and only if } s(p_i) < s(p_k) + d^+(p_i, p_k),$$

or p_k is bigger than p_i if and only if p_k and p_i were incomparable.

For case for p_l and p_j is similar and we omit the proof.

(3) If $p_i > p_k$, then $0 < s(p_i) - s(p_k) < n$. Thus $s'(p_i) - s(p_k) = s(p_i) - n - s(p_k) < 0 < d^+(p_i, p_k)$, and p_i is not bigger than p_k anymore. By (2), p_k is not bigger than p_i as well. \square

2.3. The Odd-Even Routing Algorithm. The results on the routing number of P_n were shown using what is known as the **odd-even routing algorithm**. First we describe the odd-even routing algorithm on the path. Label the vertices of P_n as v_1, v_2, \dots, v_n . We say an edge $e = v_i v_{i+1}$ is an odd edge if i is odd; otherwise i is even and e is an even edge. Note that the set of odd edges and even edges partition P_n into two maximal matchings. During the first step and every other odd step of the routing process, we consider only the odd edges. We select a subset of the odd edges and swap the pebbles on the endpoints. During the even steps of the routing process we consider only the even edges and act similarly. During each step, the edges that are selected are those where swapping the pebbles take them closer to their destinations.

We can generalize this algorithm to even cycles. It is well-known that the edges of an even cycle can be partitioned into two perfect matchings, and we would call edges in one perfect matching to be even and the others to be odd. Thus once we specify one edge to be odd (or even), we know the parity of the edges. Given a particular disbursement B , each vertex is given a particular spin. During odd steps we choose a matching of odd edges and two pebbles on edge $e_i = v_i v_{i+1}$ swap only if the spin of the pebble at vertex v_i is greater than the spin of the pebble at vertex v_{i+1} . During even steps we do the same using only even edges. In the future, if choose e to be an odd edge, we would call this algorithm to be **odd-even routing algorithm with odd edge e** .

Note that this algorithm is not defined on cycles of odd length since the edges that would be labeled as odd edges do not form a matching.

2.4. The Window of a Pebble. Let $G = C_n$ in this section, where $n \geq 3$ is an even integer. We fix a minimized disbursement B of π with associated order \succ .

When the odd-even routing algorithm is applied, we can count the number of rounds necessary for each pebble to reach and stay at its destination vertex, then the maximum of all these values is obviously an upper bound on $rt(C_n, \pi)$.

For any arbitrary pebble A , let

$$U = \{p \in P : p \succ A\} \text{ and } W = \{p \in P : A \succ p\}.$$

By Lemma 4, the routing process ends when no pebble has pebbles bigger or smaller than it, therefore $s(A) = |W| - |U|$.

By Lemma 3, there is no $u \in U, w \in W$ that are ordered as u, w, A or A, u, w along the positive direction. So if $U = \{u_1, u_2, \dots, u_r\}$ and $W = \{w_1, w_2, \dots, w_t\}$, then we may assume that the pebbles in $U \cup W$ and A are ordered along positive direction on the cycle as $u_r, \dots, u_1, A, w_1, \dots, w_t$. We denote the set of pebbles incomparable to A between A and w_t (between u_r and A resp.) by X (Y resp.).

A *segment* is a sequence of consecutive pebbles. If all the elements in a segment are from U , then we call it to be an U -segment, and similarly for W -segments, X -segments and Y -segments. So we can group the pebbles between u_r and w_t along the positive direction as

$$win(A) = (U_k, Y_k, U_{k-1}, \dots, U_1, Y_1, A, X_1, W_1, \dots, X_l, W_l),$$

where X_1, Y_1 may be empty, and $win(A)$ is called the *initial window* of A . We denote the set of all other pebbles as Z . So sometimes we write π as

$$\pi = (Z, U_k, Y_k, U_{k-1}, \dots, U_1, Y_1, A, X_1, W_1, \dots, X_l, W_l).$$

By transitivity, we have $u_i \succ w_j$ since $u_i \succ A \succ w_j$ for all $1 \leq i \leq r$ and $1 \leq j \leq t$, and in particular, $u_r \succ w_t$, hence $n \geq s(u_r) - s(w_t) > d^+(u_r, w_t)$. By Lemma 3, if $i \geq j$, then $u \succ y$ for all $u \in U_i, y \in Y_j$; If $k \geq l$, then $w \prec x$ for all $w \in W_k, x \in X_l$.

3. CONSECUTIVE MOVES AND ROTATION PERMUTATIONS

Lemma 7. *Let p be a pebble and Q be a segment of pebbles and $p \succ Q$ (or $Q \succ p$), then once p starts to swap with a pebble in Q , p will not stop swapping until p swaps with all pebbles in Q (in the following $|Q| - 1$ or more steps).*

Proof. If the pebbles in Q that have yet to swap with p remains to be a segment (with a different order or not) in the routing process, then it is clear that p would swap with Q consecutively. Also note that if some pebble smaller than a pebble in Q is between pebbles in Q , this pebble is also smaller than p by transitivity thus will not delay the movement of p . Similarly for a pebble smaller than p . We should call it an *enlargement* of Q if a smaller pebble (than p) is mixed with Q . If Q' is an enlargement of Q , then $p \succ Q'$ and Q' is a segment. Note that we shall similarly consider the enlargement of the yet-to-swap-with- p pebbles in Q if necessary.

Consider the initial window of p and let the segments between p and Q be

$$p, X_1, W_1, \dots, X_k, W_k = Q,$$

where pebbles in W_i with $i \leq k$ are all smaller than p , and pebbles in X_i with $i \leq k$ are incomparable with p (thus by Lemma 3, are bigger than pebbles in Q). We claim that at most one pebble from $X = \cup_{i=1}^k X_i$ is between any consecutive pair of pebbles in an enlargement of Q (note that if a pebble $u \succ p$ and swapped with p , then we consider u as an incomparable

pebble with p thus in X as well). For otherwise, let step s be the first step so that some pebbles $x, x' \in X$ and q, q' in an enlargement of Q are ordered consecutively as q, x, x', q' . Since this is the first step, qx and $x'q'$ were edges to be swapped in the previous step. But the edge qx would have swapped q and x , and give x, q, x', q' in step s , a contradiction.

Assume that p swaps with $q_1 \in Q$ in step s , and first stop is at step t before finish swapping with Q . Then before the step $t-1$, the pebbles following p are z_1, z_2, z_3, z_4 with $p \succ z_1$. Note that at most one of z_2 and z_3 is from X , so at least one of them is smaller than p . At step $t-1$, pz_1, z_2z_3 are the edges to swap; if $z_2 \in X$ then $z_3 \notin X$ thus $p \succ z_3$, therefore after the swap we have z_1, p, z_3, z_2 and pz_3 swap at step t ; and if $z_2 \notin X$, then $p \succ z_2$, and after the swap, we either have z_1, p, z_3, z_2 or z_1, p, z_2, z_3 , the former case occurs if $z_2 \succ z_3$ thus $p \succ z_3$ so p, z_3 swap in step t , and in the latter case p, z_2 swap in step t . \square

Lemma 8 (Rotation Lemma). *Suppose π is a rotation permutation such that $\pi(a) = a + q \pmod{n}$ for some integer q , where $-\frac{n}{2} < q \leq \frac{n}{2}$. Then, $rt(C_n, \pi) = n - |q|$.*

Proof. By symmetry we only consider the case when $q > 0$. We first show $rt(C_n, \pi) \geq n - q$. For each pebble p , the spin of p is either $n - q$ or $-q$. Since the sum of spins is zero there must be exactly q pebbles with positive spin and $n - q$ pebbles with negative spin. So, $n - q \leq rt(C_n, \pi)$.

Now, we show $rt(C_n, \pi) \leq n - q$. We order clockwise as p_1, p_2, \dots, p_n so that p_{2i-1} with $1 \leq i \leq q$ have positive spin $n - q$ and the remaining $n - q$ pebbles have negative spin $-q$. We should use an odd-even routing algorithm so that p_1p_2 is an odd edge. As $q \leq n/2$, no two pebbles with spin $n - q$ are adjacent. In the routing process, p_1 will be paired with $p_2, p_4, \dots, p_{2q}, p_{2q+1}, \dots, p_n$ in the first $n - q$ steps, thus reach its destination, and similarly, for all other pebbles with positive spins. As all pebbles with positive spins reach their destinations, there is no order relation left, so every pebble will be in place. So it is routed in $n - q$ steps. \square

4. EXTREMAL WINDOWS

Now let us consider the routing process for arbitrary pebble A . As defined, let U_1, U_2, \dots, U_k and W_1, W_2, \dots, W_l be the comparable segments with A . In the routing process, those segments may be mixed and A may not swap with them in their initial order. However, by Lemma 7, if A starts to swap with a pebble in one segment Q , then A will swap with pebbles in the following $|Q|$ steps (not necessary with elements in Q though). However, if A starts to swap with W_i , and in the following $|W_i|$ steps, A will swap with pebbles that are smaller than A , thus must be W as well; we would call those pebbles to be W'_i . So we know $|W'_i| = |W_i|$ and they both contain pebbles smaller than A . Similarly we define U'_i for $1 \leq i \leq k$. Note that U' and W' are not necessarily segments any more. So A will meet U -sequences in the order of U'_1, U'_2, \dots, U'_k , and W -sequences in the order of W'_1, W'_2, \dots, W'_l , but may meet those sequences in a mixed order. Let Z_1, Z_2, \dots, Z_{k+l} be the order the sequences meet A , so $Z_i \in \{U'_1, \dots, U'_k, W'_1, \dots, W'_l\}$.

By Lemma 7 and the above definition, once A meets Z_i , A would swap with Z_i in the following $|Z_i|$ steps, but A may wait to meet Z_{i+1} after finishing swapping Z_i . For $i = 1, 2, \dots, k + l - 1$, let ω_i be the number of steps A waits between swapping with the last pebble of Z_{i-1} and the first pebble of Z_i . We call ω_i the waiting time between Z_{i-1} and Z_i . We get $k + l$ nonnegative umbers $\omega_i, i = 1, 2, \dots, k + l$.

Now suppose α is the largest index such that $\omega_\alpha \neq 0$. First we assume that Z_α is the t -th W -sequence. Note that if a swap of A and W (or A and U) cannot be followed by a swap of A and U (or A and W) because of the parity. So as $\omega_{\alpha+1} = \dots = \omega_{k+l} = 0$, A will swap with $\sum_{j=t}^l |W'_j|$ W -pebbles in the following steps without stop until it arrives its destination.

Let w' be the first pebble A meets in $W'_t = Z_\alpha$. Then

(i) As $\omega_\alpha > 0$, W'_t will not merge with another W -sequence in the routing process before encountering A , thus some pebble in W'_t is always paired with an X -pebble in the routing process from the first step or the second step according to parity of edges, therefore moves in the counter clockwise direction after that. So w' could be the one of the two leftmost pebbles in W'_t .

(ii) A begin to swap with Z_α if and only if w' has swapped with all the X -pebbles in $\cup_{j=1}^t X_j$ and all the U -pebbles in $\cup_{i=1}^k U_i$.

Thus the total number of steps needed for A to be in place must be:

$$\sum_{j=1}^t |X_j| + \sum_{j=t}^l |W_j| + \sum_{i=1}^k |U_i| + \delta,$$

where $\delta = 0$ if w' is paired with an X -pebble in the first step, otherwise $\delta = 1$.

For pebble A to be sorted within $n - 1$ steps, we have

$$\sum_{i=1}^k |U_i| + \sum_{j=t}^l |W_j| + \sum_{j=1}^t |X_j| + \delta \leq n - 1.$$

Notice that $O + \sum_{i=1}^k (|U_i| + |Y_i|) + \sum_{j=1}^l (|W_j| + |X_j|) = n - 1$, where O is the number of pebbles outside the range of $\text{win}(A)$. Then we have

$$(n - 1) - \sum_{j=t+1}^l |X_j| - \sum_{i=1}^k |Y_i| - \sum_{j=1}^{t-1} |W_j| + \delta - O \leq n - 1.$$

which implies that every permutation that takes $n - 1$ steps to route must contain a pebble A such that

$$(1) \quad \sum_{j=1}^k |Y_j| + \sum_{j=t+1}^l |X_j| + \sum_{j=1}^{t-1} |W_j| + O = \delta, \text{ where } \delta \in \{0, 1\}.$$

Similarly, if $Z_\alpha = U'_t$ and $u' \in U'_t$ is the first pebble A meets in U'_t , then the total number of steps needed for A to be in place must be:

$$\sum_{j=1}^t |Y_j| + \sum_{j=t}^k |U_j| + \sum_{i=1}^l |W_i| + \delta,$$

where $\delta = 0$ if u' is paired with an Y -pebble in the first step, otherwise $\delta = 1$, and it follows that every permutation that takes $n - 1$ steps to route must contain a pebble A such that

$$(2) \quad \sum_{j=1}^l |X_j| + \sum_{j=t+1}^k |Y_j| + \sum_{j=1}^{t-1} |U_j| + O = \delta, \text{ where } \delta \in \{0, 1\}.$$

Lemma 9. *Every permutation that takes $n - 1$ steps to route must contain a pebble A whose windows is one of the following*

- (1) $|win(A)| = n$ and $win(A) = (A, X, W)$ (or $win(A) = (U, Y, A)$).
- (2) $|win(A)| = n - 1$ and $win(A) = (U, A, X, W)$ (or $win(A) = (U, Y, A, W)$).
- (3) $|win(A)| = n$, and $win(A) = (A, X_1, W_1, X_2, W_2)$ and $\min(|W_1|, |X_2|) = 1$ (or $win(A) = (U_2, Y_2, U_1, Y_1, A)$ and $\min(|Y_2|, |U_1|) = 1$).

Proof. By symmetry, we may assume that (1) holds. As $\delta = 0$ or 1, all the terms in the left-hand side of (1) are zero or one.

If $O = 0$ and $\delta = 0$, then all terms are zero and $|win(A)| = n$. Note that $\sum_{j=1}^k |Y_j| = 0$ implies that $Y = \emptyset$, thus there is at most one U -set; $\sum_{j=t+1}^l |X_j| = 0$ means there are at most t non-empty X -sets (the first t sets); $\sum_{j=1}^{t-1} |W_j| = 0$ means that $|W| = 0$ (if $t \geq 2$) or there are at most one non-empty W -set (that is, W_1 when $t = 1$). But if $t = 2$, $|W| = 0$ implies all X -sets are actually outside of window of A , so $|X| = 0$ as well. Therefore, $win(A) = (U, A, X, W)$ so that if $W = \emptyset$ then $X = \emptyset$.

If $\delta = 1$ and $O = 1$, then it is the same as above, except that $|win(A)| = n - 1$. So $win(A) = (U, A, X, W)$ and $|win(A)| = n - 1$.

If $\delta = 1$ and $O = 0$, then $|win(A)| = n$, and one of the first three terms on the left-hand side is one.

If $\sum_{j=1}^k |Y_j| = 1$, then $|Y| = 1$, so there are at most two U -sets, U_1 and U_2 , and when there are two, $Y_1 = \emptyset$ and $|Y_2| = 1$. Together with $\sum_{j=t+1}^l |X_j| = 0$ and $\sum_{j=1}^{t-1} |W_j| = 0$, we have $win(A) = (U, A, X, W)$, (U_2, y, U_1, A) or (U_2, y, U_1, A, X, W) .

If $\sum_{j=1}^k |Y_j| = 0$, $\sum_{j=t+1}^l |X_j| = 1$ and $\sum_{j=1}^{t-1} |W_j| = 0$, then $Y = \emptyset$ and $t = 1$ and $|X_2| = 1$. So $win(A) = (U, A, X_1, W_1, x_2, W_2)$.

If $\sum_{j=1}^k |Y_j| = 0$, $\sum_{j=t+1}^l |X_j| = 0$ and $\sum_{j=1}^{t-1} |W_j| = 1$, then $Y = \emptyset$, $t = 2$ and $|W_1| = 1$, and $X_i = \emptyset$ for $i \geq 3$. So $win(A) = (U, A, X_1, w_1, X_2, W_2)$.

However, when $|win(A)| = n$, we claim that one of U_k and W_l must be empty. For otherwise, let $u_p \in U_k$ and $w_q \in W_l$ be the furthest U -pebble and W -pebble to A , respectively. As $|win(A)| = n$, no pebble is bigger than u_p and no pebble is smaller than w_q , so $s(u_p) \geq 1 + |Y| + |W|$ and $s(w_q) \leq -(1 + |X| + |U|)$, and it follows that $s(u_p) - s(w_q) \geq n + 1$, a contradiction.

So we have the desired extremal windows in the lemma. \square

5. THE PROOF

In this section, we show how to deal with the extremal situations in Lemma 9.

We will consider the structures of the windows more specifically, and the following concepts are useful. Let q_1, q_2, \dots, q_s be a sequence of consecutive pebbles. It is called a *block with head* q_1 if the only order relation among them are $q_1 \succ q_i$ for $i \geq 2$; it is called a *block with tail* q_s if the only order relation among them are $q_i \succ q_s$ for $i < s$; it is called an *isolated block* if there is no order relation among them.

5.1. Extremal window type I: $win(A) = (A, X, W)$ and $|win(A)| = n$.

Lemma 10. *If permutation π needs $n-1$ steps to route and some pebble A in π has $win(A) = (A, X, W)$ and $|win(A)| = n$, then π is $(123\dots)$ or its inverse.*

Proof. Let $X = x_1 x_2 \dots x_a$ and $W = w_1 w_2 \dots w_b$ in the clockwise order. Consider the spins of A and w_b . Note that $spin(A) = |W|$ and $s(w_b) \leq -(1 + |X|)$ since no pebble is smaller than

w_b (by Lemma 3) and w_b needs to swap with A and all the pebbles in X . So $s(A) - s(w_b) \geq n$ and it follows that $s(w_b) = -1 - |X|$ and w_b only swaps with the pebbles in $X \cup A$. Now again by Lemma 3, no pebble is smaller than w_{b-1} and use the same argument, we have $s(w_{b-1}) = -1 - |X|$ and inductively $s(w_i) = -1 - |X|$ for $1 \leq i \leq b$.

Consider the spin of x_1 , it is clear that $s(x_1) \geq |W|$ since x_1 need to swap with all pebbles in W and no pebble is bigger than x_1 (analogously to Lemma 3), then $s(x_1) - s(w_b) \geq n$ and it follows that $s(x_1) = |W|$ and $x_1 \succ W$ is the only order relation involving x_1 . Inductively we have $s(x_i) = |W|$ for all $x_i \in X$ and $\{A\} \cup X \succ W$ is the only order relation in the permutation.

So along positive direction every pebble is $|W|$ steps away from its destination. So π is a rotation. By the Rotation Lemma 8, the only rotations that require $n - 1$ steps to route are $\pi = (123 \cdots n)$ or its inverse. \square

5.2. Extremal window type 2: $\text{win}(A) = (U, A, X, W)$ and $|\text{win}(A)| = n - 1$.

Lemma 11. *If a permutation π contains a pebble A such that $\text{win}(A) = (U, A, X, W)$ and $|\text{win}(A)| = n - 1$ and $\pi = (z, U, A, X, W)$, then U and W are isolated blocks and X can be decomposed into isolated blocks and blocks with tails. Furthermore, $s(z) = c \leq 0$, and if $c < 0$, then the block of X next to W , say X_0 , is an isolated block with $-c$ pebbles, and the only other order relation is $U \cup \{A\} \cup X \succ W$ and $X_0 \succ \{z\}$.*

Proof. Let $U = u_1 u_2 \dots u_p$, $X = x_1 x_2 \dots x_a$ and $W = w_1 w_2 \dots w_b$ along the positive direction. Consider the spins of u_1 and w_b . As no pebble is bigger than u_1 and u_1 is bigger than A and W , $s(u_1) \geq 1 + |W| = 1 + b$. Similarly, no pebble is smaller than w_b and w_b is smaller than U, A, X , so $s(w_b) \leq -(1 + |U| + |X|) = -(1 + a + p)$. So $s(u_1) - s(w_b) \geq 1 + b + 1 + a + p = n$, and equalities must hold. So $s(u_1) = 1 + b$, $s(w_b) = -(1 + a + p)$ and the only order relation involving u_1 and w_b are $u_1 \succ \{A\} \cup W$ and $U \cup \{A\} \cup X \succ w_b$. Inductively we can consider u_2 and w_{b-1} and all pebbles in U and W and conclude that $U \cup \{A\} \cup X \succ W$ is the only order relation involving U and W .

Now consider the spins of pebbles in X . As $s(w_b) = -(1 + a + k)$ and $s(x) - s(w_b) \leq n$ for each $x \in X$, we have $s(x) \leq b + 1$. Note that z cannot be bigger than any pebble in X , for otherwise $z \succ W$ and contradict to what we just concluded. But z may be smaller than some pebbles in X , thus $s(z) \leq 0$.

Consider x_1 . As no pebble is bigger than x_1 (again analogously to Lemma 3) and $x_1 \succ W$, $s(x_1) \geq |W| = b$. So $s(x_1) \in \{b, b + 1\}$. If $s(x_1) = b$, then $x_1 \succ W$ is the only order relation involving x_1 , and we will inductively consider x_2 . If $s(x_1) = b + 1$, then x_1 is bigger than (only one) another pebble besides W , either $x_i \in X$ or z ; if $x_1 \succ z$, then $x_j \succ z$ for $1 \leq j \leq a$ by Lemma 3 and we can inductively conclude $s(x_j) = b + 1$, thus X is an isolated block and $X \succ z$; if $x_1 \succ x_i$ for some $2 \leq i \leq a$, then $x_1 \succ x_j$ for $1 \leq j < i$ by Lemma 3, and no other pebble in X is smaller than x_i , for otherwise it would be smaller than x_1 which contradicts what we just concluded. So $x_1 x_2 \dots x_i$ is an block with head x_1 . Now we similarly consider x_{i+1} and get the block partition of X . \square

Now we are ready to show that such permutations can be routed in $n - 2$ steps.

Lemma 12. *If a permutation π contains a pebble A such that $\text{win}(A) = (U, A, X, W)$ and $|\text{win}(A)| = n - 1$ and $\pi = (z, U, A, X, W)$, then π can be routed in at most $n - 2$ steps.*

Proof. First we assume that $X \neq \emptyset$. Let $\pi = zu_1 \dots u_k Ax_1 \dots x_a w_1 \dots w_b$ in the clockwise order, with $u_i \in U, x_i \in X$ and $w_i \in W$. We use an odd-even routing algorithm so that $x_a w_1$ is an odd edge. We will make use of the structure in Lemma 11.

By Lemma 7, x_a swaps with w_1 in the first step, thus swaps with W in the following $|W| - 1$ steps, so w_b meets (i.e., is paired with a pebble in) $U \cup \{A\} \cup X$ after $|W| - 1$ steps, then w_b would swap with $U \cup A \cup X$ in the following $|U \cup \{A\} \cup X|$ steps, so it takes $|W| - 1 + |U \cup \{A\} \cup X| = n - 2$ steps for w_b to be in place. As a pebble in $U \cup \{A\} \cup X$ has to pass $W - w_b$ to meet w_b , all pebbles in W would be in place after $n - 2$ steps.

Similarly, w_1 swaps with a pebble in $X \cup U \cup \{A\}$ in the first step, it will swap with them in the following $|X \cup U \cup \{A\}|$ steps. So w_1 would have met A in $|X \cup U|$ steps and in the meantime, A has swapped with U , in other words, A has swapped with U and meets W after $|X \cup U|$ steps, and it takes $|W|$ steps for A to swap W , so A will be in place after $|X| + |U| + |W| = n - 2$ steps.

As A and W are in places after $n - 2$ steps, U are in places after $n - 2$ steps, as the only order relation on U is $U \succ \{A\} \cup W$.

So now we only need that all other order relations are taken care within $n - 2$ steps. A tail x_i in block $X' \subseteq X$ is paired with its block in the first (if $x_{i-1}x_i$ is an odd edge) or the second (if $x_{i-1}x_i$ is an even edge and $x_i \neq x_a$ or $|W| + 1$ -th step (if $x_i = x_a$); in either case, x_i would be swapping with its block or W in the next $|X'| + |W|$ steps, so will be in place after at most $n - 2$ steps. As x_1 meets W in the first step, x_1 swaps with W in the next $|W| - 1$ steps, thus meets z after $|W|$ steps, and after that, z swaps with the isolated block bigger than it, in at most $|s(z)| \leq |X|$ steps, so z would be in place after $|W| + |X| < n - 2$ steps.

Now we assume $X = \emptyset$ and $\pi = zu_1 \dots u_k Aw_1 \dots w_b$ in the clockwise order. By Lemma 11, $s(u) = 1 + b$ and $s(w) = -1 - k$ for $u \in U, w \in W$ and the only order relation is $U \cup \{A\} \succ W$. We first flip the spins of u_1 and w_b . Then the order relations are $U - u_1 \succ \{A\} \cup (W - w_b)$, $w_b \succ (W - w_b) \cup \{z, u_1\}$ and $(U - u_1) \cup \{z, w_b\} \succ u_1$ by lemma 6. We use an odd-even routing algorithm so that Aw_1 is an odd edge.

Similar to above, u_k meets $W - w_b + A$ in the second step thus swaps with them all in the following $|W|$ steps, in other words, A meets u_k thus $U - u_1$ after $|W|$ steps, by which A has swapped with $W - w_b$. So it takes $|W| + |U| - 1 = n - 3$ steps for A to be in place.

As w_1 meets $U - u_1 + A$ in the first step, it will swap with them in the following $|U|$ steps, thus w_1 meets u_2 in $|U| - 1$ steps, or u_2 meets $W - w_b + A$ after $|U| - 1$ steps, then u_2 will swap with $W - w_b + A$ in the following $|W|$ steps; meanwhile, u_1 meets $\{z, w_b\}$ in the first or second step, it takes up to three steps for u_1 to swap with z and w_b ; so u_2 will meet u_1 in the $\max\{|U| - 1 + |W|, 3\}$ -th step; as $n = |W| + |U| + 2 \geq 6$, u_2 swaps with u_1 after $|U| - 1 + |W|$ steps, so u_2 will be in place after $|U| + |W| = n - 2$ steps. As u_2 is the furthestest pebble to u_1 along the negative direction, all pebbles in $U - u_1$ will be in places after $n - 2$ steps. It also follows that u_1 is in place after $n - 2$ steps, as the only order relations involving u_1 are $(U - u_1) \cup \{z, w_b\}$.

To show every pebble to be in places after $n - 2$ steps, we just need to further show that z and w_b will be in places after $n - 2$ steps, because all the remaining order relations involve them.

As z is paired one of the u_1 and w_b in the first step, and the other in the third step, z will be in place in $4 \leq n - 2$ steps. As shown above, w_1 swaps with $U - u_1 + A$ in the first $|U|$ steps, and w_b swaps with z and u_1 in the first three steps, so w_b meets w_1 in at most

$\max\{|U|, 3\}$ steps, and swap with $W - w_b$ in the following $|W| - 1$ steps, therefore w_b will be in place after $\max\{|U|, 3\} + |W| - 1$ steps, and it is at most $n - 2$ if $|U| \geq 2$, so the only case we are in trouble is when $|U| = 1$.

But in this trouble case, instead of let Aw_1 be an odd edge, we will let u_1A be odd, then we will not have trouble unless $|W| = 1$ by symmetry, however it follows that $|U| = |W| = 1$ and $n = 4$, a contradiction to $n \geq 6$. \square

5.3. Extremal window type 2a: $\text{win}(A) = (A, X, W)$ and $|\text{win}(A)| = n - 1$. This is the case of $\text{win}(A) = (U, A, X, W)$ with $U = \emptyset$. In this case, the spins in W are not fixed anymore, so X or W could have some freedom on their spins, but only one of them could have a block decomposition. More specifically, we have the following structure lemma.

Lemma 13. (*block decomposition*) *If permeation π has a pebble A with $\text{win}(A) = (A, X, W)$ and $|\text{win}(A)| = n - 1$ and $\pi = (z, A, X, W)$, then one of the following must be true*

- (1) *if $s(z) = c > 0$, then X is an isolated block and W can be partitioned into isolated blocks and blocks with heads so that the block next to X is isolated with c pebbles and smaller than z , and the only other order relation is $\{A\} \cup X \succ W$.*
- (2) *if $s(z) = c < 0$, then W is isolated and X can be partitioned into isolated blocks and blocks with tails so that the block next to W is isolated with $|c|$ pebbles and bigger than z , and the only other order relation is $\{A\} \cup X \succ W$.*
- (3) *if $s(z) = 0$, then either X can be partitioned into isolated blocks and blocks with tails and W is an isolated block or W can be partitioned into isolated blocks and blocks with heads and X is an isolated block, and the only other order relation is $\{A\} \cup X \succ W$.*

The proof of this lemma is very similar to Lemma 11, and for completeness, we include a proof below.

Proof. Let $\pi = zAx_1x_2 \dots x_aw_1w_2 \dots w_b$ along the positive direction, where $x_i \in X$ and $w_i \in W$. Clearly, $s(A) = |W| = b$. As z is incomparable with A , no pebble in W is bigger than z (but could be smaller than z), thus z could only cause pebbles in W move one step along the negative direction. Similarly, no pebble in X is smaller than z (but could be bigger than z), thus z could only cause pebbles in X move one step along the positive direction.

For $w \in W$, $|s(w)| + |s(A)| \leq n$, thus $|s(w)| \leq n - b = a + 2$. Consider w_b . As no pebble is smaller than w_b by Lemma 3, $s(w_b) < 0$ thus $s(w_b) \leq -(|X| + 1) = -(a + 1)$. Therefore $s(w_b) \in \{-(a + 1), -(a + 2)\}$, and $s(w_b) = -(a + 2)$ if and only if w_b is smaller than z or some pebble in W but not both. If w_b is not smaller than z and any other pebbles in W , then we call w_b to be *isolated* and in this case $s(w_b) = -(a + 1)$.

If w_b is smaller than w_i , then w_j with $i < j < b$ is not comparable with w_b as w_b can only have one bigger pebble in W , then must be smaller than w_i by Lemma 3. Note that no pebble w_l with $l < i$ could be bigger than w_i , as otherwise it would be bigger than w_b which is impossible. Now we can inductively conclude that w_{b-1}, \dots, w_{i+1} all have spin $-(a + 2)$ and smaller than $\{A, w_i\} \cup X$. That is, $\{w_i, w_{i+1}, \dots, w_b\}$ is a block with head w_i .

If w_b is smaller than z , then $s(w_b) = -(a + 2)$ and w_b is not comparable to any other pebbles in W . Furthermore, by Lemma 3 w_j with $1 \leq j < b$ must be smaller than z as well, thus inductively we can conclude that w_{b-1}, \dots, w_1 all have spin $-(a + 2)$ and (only) smaller than $\{z, A\} \cup X$. In this case $\{w_1, \dots, w_b\}$ is an isolated block which is smaller than z .

If w_b is in a block with head w_i , then we can repeat the above argument of w_b for w_{i-1} and conclude that w_{i-1} is in a block, or isolated. Inductively we may partition W into blocks

W_0, W_1, \dots , that are isolated or with heads. In particular, if $z \succ w_i$, then w_i must be in an isolated block, and by Lemma 3, $z \succ \{w_1, w_2, \dots, w_i\}$, that is, $w_1 w_2 \dots w_i$ is an isolated block.

If one pebble from W has spin $-(a+2)$, then the spins of pebbles in X are all b (by inductively consider x_1, x_2, \dots, x_a), and they are incomparable to each other and they all bigger than W , so X is an isolated block. This means also that if $s(z) = c \geq 0$, then the isolated block $w_1 w_2 \dots w_i$ has c pebbles.

Similarly if one pebble from X has spin $b+1$, then W is an isolated block and X can be partitioned into isolated blocks and blocks with tails, and in particular, the isolated block bigger than z has $|s(z)|$ pebbles. \square

Lemma 14. *If a permutation π contains a pebble A with $\text{win}(A) = (A, X, W)$ and $|\text{win}(A)| = n-1$, then π can be routed in at most $n-2$ steps or is $(12 \dots n)$ or its inverse.*

Proof. We only consider the case $s(z) = c \geq 0$. By Lemma 13, we assume that X is an isolated block, and W has the block decomposition W_0, W_1, \dots, W_k so that the block W_i , if not isolated, has head w_i and W_0 is an isolated block with c pebbles which are all smaller than z .

If $c = b$, then the spins of $\{z, A\} \cup X$ are all b and the spins of X are all $-(a+2)$, and π is a rotation. Thus if π needs $n-1$ steps to route, π must be one of the two extremal permutations. So we assume $c < b$.

If $a = |X| = 0$, then we use the odd-even routing algorithm so that Aw_1 is an odd edge. By Lemma 7, A meets W in the first step and will swap with W in the following $|W| = n-2$ steps, and z meets W_0 in the second step, so will swap with W_0 in $c+1 < b+1 = n-1$ steps. For a head w of the block W_i , it will swap with its block either in the first step, or the second step (if on an even edge) or the third step (if on an even edge and meets A in the first step); for the first case, the head swaps with all pebbles in W_i and A in $|W_i| - 1 + 1 = |W_i| \leq n-2$ steps; for the second case, $|W_i| \leq n-3$ and the head swaps with all pebbles in W_i and A in $(|W_i| - 1) + 1 + 1 = |W_i| + 1 \leq n-3 + 1 = n-2$ steps; for the third case, $|W_i| \leq n-2$ and w swaps with all pebbles in W_i and x in $|W_i| - 1 + 2 = |W_i| + 1 \leq n-1$ steps. So every pebble will finish their swaps in at most $n-2$ steps, except $\pi = (z, A, w, w_1, \dots, w_{n-3})$, where $s(z) = 0$ and w is the head the block $ww_1 \dots w_{n-3}$. For the exceptional case, we flip the spins of A and w_{n-3} , by Lemma 6, w_{n-3} is bigger than $z, A, w_1, \dots, w_{n-4}$, and A is smaller than z and w_{n-2} and $ww_1 \dots w_{n-4}$ remains to be a block with head w . Now we use the odd-even sorting algorithm so that $w_{n-3}z$ is an odd edge. By Lemma 7, w_{n-3} swaps in the first $n-2$ steps, w swaps from the second step and takes $n-4$ steps, A steps in 3 steps, and z swaps in 3 steps, and in at most $n-2$ steps all pebbles will be in their places. Therefore we may assume that $a > 0$.

Now we use the odd-even routing algorithm so that $x_a w_1$ is an odd edge. By Lemma 7, x_i with $1 \leq i \leq a$ meets W after $a-i$ steps and swaps with W in the following $|W|$ steps, so it takes $a-i+|W| = a+b-i = n-2-i \leq n-2$ steps; A could be regarded as x_0 , so takes at most $n-2$ steps; z meets W_1 after $a+1$ steps and takes c swaps, so will be in place in at most $a+1+c < a+b+1 \leq n-2$ steps; the head w in block W_i swaps with W_i at the first step, or the second step (if it is on an even edge and not adjacent to x), or after $a+2$ steps (if $w = w_1$), and in the former two cases it takes at most $1+(|W_i|-1)+a+1 = a+|W_i|+1 \leq a+b = n-2$ steps, and in the last case, it takes $a+2+(|W_i|-1) = a+|W_i|+1 \leq n-2$ steps as well. Once all of the above pebbles are in place, all pebbles are in place as well, as no swap remains. \square

5.4. Extremal window type 3: $|win(A)| = n$ and $win(A) = (A, X_1, W_1, x, W_2)$ or $win(A) = (A, X_1, w, X_2, W_2)$.

Lemma 15. (*block decomposition*) *If a permutation π contains a pebble A with $win(A) = (A, X_1, W_1, x, W_2)$ or $win(A) = (A, X_1, w, X_2, W_2)$ and $|win(A)| = n$, then X_1 and W_2 are isolated blocks and*

- *if $win(A) = (A, X_1, W_1, x, W_2)$, then W_1 can be partitioned into isolated blocks and blocks with heads. Furthermore, $c := s(x) - |W_2| \geq 0$, and if $c > 0$, then the block W_0 in W_1 next to X_1 is an isolated block with c elements and are all smaller than x ; and the only other order relation between segments are $\{A\} \cup X_1 \succ W_1 \cup W_2, x \succ W_2$ and $z \succ W_0$.*
- *if $win(A) = (A, X_1, w, X_2, W_2)$, then X_2 can be partitioned into isolated blocks and blocks with tails. Furthermore, $c := s(w) + (1 + |X_1|) \leq 0$, and if $c < 0$, then the block X_0 in X_2 next to W_2 is an isolated block with $-c$ elements and are all bigger than w ; and the order relation between segments are $\{A\} \cup X_1 \succ W_1 \cup W_2$ and $X_0 \succ w$.*

Proof. We only prove the case when $win(A) = (A, X_1, W_1, x, W_2)$, and the other case is symmetric. Let $W_1 = w_1 w_2 \dots w_a$ and $W_2 = w_{a+1} w_2 \dots w_b$. Consider the spins of A and w_b . Since $s(A) = b$ and no pebble is smaller than w_b thus $s(w_b) \leq -(2 + |X|) = -(n - b)$; furthermore, as $s(A) - s(w_b) \leq n$, we have $s(w_b) \geq -(n - b)$, so $s(w_b) = -(n - b)$, and the only pebbles that w_b swaps with are A and all x 's. Since w_{b-1} does not cross with w_b , we get $s(w_{b-1}) \leq -(n - b)$. For the same reason, we have $s(w_{b-1}) = -(n - b)$. By induction, we have $s(w_i) = -(n - b)$ for all $w_i \in W_2$. Similarly, by comparing the spins of pebbles in X_1 to that of w_b , we have $s(x_i) = b$ for all $x_i \in X_1$. Since $s(A)$ is the same as the pebbles in X_1 , let $X'_1 = X_1 \cup \{A\}$. We have shown that X_1 and W_2 are isolated blocks.

Now we consider the spins of pebbles in W_1 .

We note that no pebble in W_1 is bigger than x , for otherwise A would be bigger than x as $A \succ W_1$. We should also note, however, that x could be bigger than some pebbles in W_1 . Consider the pebble $w_a \in W_1$. The pebbles in X'_1 are bigger than w_a and no pebble is smaller than w_a , so $s(w_a) \leq -|X'_1| = b - n + 1$. On the other hand, $s(A) - s(w_a) \leq n$ and $s(A) = b$ implies $s(w_a) \geq b - n$. That is, $s(w_a) \in \{b - n, b - n + 1\}$, and at most one pebble other than those in X'_1 is bigger than w_a .

If $s(w_a) = b - n + 1$, then w_a is incomparable with pebbles other than those in X'_1 . If $s(w_a) = b - n$, then w_a is smaller than x or some pebble $w_i \in W_1$; in the former case, all pebbles in W_1 are smaller than x and inductively one can show that they are incomparable and thus W_1 is an isolated block; in the latter case, $w_i \succ w_j$ for $i + 1 \leq j \leq a$ and w_i is the only such pebble other than those in X'_1 , so $w_i w_{i+1} \dots w_a$ is a block with head w_i . Inductively one can have a partition of W_1 into blocks, as desired.

We observe that if $x \succ w_i \in W_1$, then w_i is in an isolated block and $x \succ \{w_1, w_2, \dots, w_i\}$ by Lemma 3. Let $c := s(x) - |W_2|$, then $W_0 := (w_1, \dots, w_c)$ is an isolated block, $x \succ W_2 \cup W_0$ are only orders relative to x , as desired. \square

Lemma 16. *If a permutation π contains a pebble A such that $|win(A)| = n$ and $win(A) = (A, X_1, W_1, x, W_2)$ or $win(A) = (A, X_1, w, X_2, W_2)$, then π can be routed in $n - 2$ steps.*

Proof. Again we only consider the case $win(A) = (A, X_1, W_1, x, W_2)$, as the other one is very similar. Let $win(A) = (Ax_1 x_2 \dots x_k w_1 \dots w_a x w_{a+1} \dots w_b)$. Before the routing, we flip the spins of A and w_b . Now by Lemma 6 and Lemma 15, the blocks in W_1 and X_1 remain the same, $W_2 - \{w_b\}$ is an isolated block, $X_1 \cup \{w_b\} \succ W_1 \cup (W_2 - w_b) \cup \{A\}$, and $x \succ W_2 - w_b$.

We will use an odd-even routing algorithm so that $x_k w_1$ is an odd edge. By Lemma 7, A and w_b will meet in the first or second step depending on whether it is on an even or odd edge, thus after the first two steps, we may think w_b is part of X_1 and A is part of the new W_2 ; Again by the lemma, x_k meets W_1 in the first step and will swap with $|W_1|$ elements in the next $|W_1|$ steps, and by then it will meet $W_2 - w_b + A$ and swap with all of them, so it takes $|W_1| + |W_2| = b < n - 1$ steps for x_k to be in place; Similarly, x_i for $1 \leq i \leq k - 1$ meets W_1 in the $k - i + 1$ -th step and swap with them and later W_2 in the following steps, so it takes $k - i + 1 + b \leq k + b = n - 2$ steps; w_b will swap with A in the first or second step and meet W_1 after all of pebbles in X_1 so in the $k + 1$ -th step, and then swap with $W_1 \cup (W_2 - w_b)$ consecutively, thus it takes $k + 1 + b - 1 = n - 2$ steps for it to be in place; for a head $w \in W' \subseteq W_1$, if it is not w_1 , then it meets the pebbles in W' in the first or second step based on the parity of the edge, and then swap with them all in the following steps by the time when W' first meets X_1 , so it adds no extra swap steps. If w_1 is a head of a block, then w_1 and x are incomparable by Lemma 15, and w_1 will meet its block (but not paired immediately) after swapping with $X_1 \cup \{w_b\}$, so it takes $|X_1 \cup \{w_b\}| + 1 + a - 1 \leq (k + 1) + 1 + (b - 1) - 1 = k + b = n - 2$ steps to be in place.

Lastly, we consider x . Assume that $s(x) = c \geq 0$. Note that x meets W_2 in the first or second step based on the parity of edge xw_{a+1} , and swap with $W_2 - w_b$ and A in the following steps, and it will meet the first block W_0 (note that $x \succ W_0$) in the $k + 1$ th step, so it may take an extra $k + 1 - (b - a - 1)$ (if xw_{a+1} is an odd edge) or $(k + 1) - (b - a - 1 + 1)$ (if xw_{a+1} is an even edge) steps for x to meet W_0 and then another c steps to swap, therefore it takes

$\max\{1 + (b - a - 1), k + 1\} + c \leq \max\{b - a + c, k + 1 + c\} \leq \max\{b, k + 1 + a\} \leq k + 1 + (b - 1) = n - 2$ steps to be in place.

As we have taken care of all order relations in at most $n - 2$ steps, every pebble is in place in $n - 2$ steps. □

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